On explosions in heavy-tailed branching random walks

Omid Amini [*] Luc Devroye [†] Simon Griffiths [‡] Neil Olver [§]

Abstract

Consider a branching random walk on $\mathbb{R}$, with offspring distribution Z and nonnegative displacement distribution W. We say that *explosion* occurs if an infinite number of particles may be found within a finite distance of the origin. In this paper, we investigate this phenomenon when the offspring distribution Z is heavy-tailed. Under an appropriate condition, we are able to characterize the pairs (Z, W) for which explosion occurs, by demonstrating the equivalence of explosion with a seemingly much weaker event: that the sum over generations of the minimum displacement in each generation is finite. Furthermore, we demonstrate that our condition on the tail is best possible for this equivalence to occur.

We also investigate, under additional smoothness assumptions, the behaviour of M_n, the position of the particle in generation n closest to the origin, when explosion does not occur (and hence $\lim_{n\to\infty} M_n = \infty$).

1 Introduction

Our aim in this paper is to give a classification of the displacement random variables in heavy-tailed branching random walks in $\mathbb{R}$ for which explosion—a concept we will define shortly—occurs. Thus consider a branching random walk on $\mathbb{R}$. The process begins with a single particle at the origin; this particle moves to another point of $\mathbb{R}$ according to a displacement distribution W, where it gives birth to a random number of offspring, according to a distribution Z. This procedure is then repeated: the particles in a given generation each take a single step according to an independent copy of the same distribution W, and then give birth to the next generation. We consider the case where W is nonnegative (in which case the process is also called an *age-dependent* process; the displacement of a particle can also be interpreted as a birthdate). Let Γ_t be the number of particles with displacement at most t; then we say that *explosion* occurs if $\Gamma_t = \infty$ for some finite t.

Alternatively, let M_n be the displacement of the leftmost particle in the n-th generation. If the process dies out and there are no particles remaining in the n-th generation, then define $M_n = \infty$. Explosion is the event that $\lim_{n\to\infty} M_n < \infty$. Note that, since M_n is monotone, it has a limit.

[*] CNRS - DMA, École Normale Supérieure, Paris, France.
[†] School of Computer Science, McGill University, Montreal, Canada.
[‡] IMPA, Rio de Janeiro, Brazil.
[§] Department of Mathematics, MIT, Cambridge, USA.

Taking a tree view of the above process, denote by T_Z a random Galton-Watson tree with offspring distribution Z, and let Z_n be the number of children at level n. To avoid the trivial case, we assume throughout that $\mathbb{P}\{Z = 1\} < 1$. Each edge of T_Z is then independently given a weight according to the nonnegative distribution W. The connection to the above process is that the displacement of a node is simply the sum of the weights on the path from the root to that node. From this perspective, which is the one we will take in this paper, explosion is the event that there exists an infinite path for which the sum of the weights on the path is finite.

In the process of studying the event of explosion, we first consider the case where the offspring distribution has finite mean. The different cases described in the next paragraph show that we can either trivially solve the problem, or reduce to the most interesting case of an infinite mean.

Reduction to the case of an infinite mean. Consider a Galton-Watson process with offspring distribution Z satisfying $0 < \mathbb{E}\{Z\} < \infty$. We still assume $\mathbb{P}\{Z = 1\} < 1$. Let W be a weight (or displacement) distribution on the edges of the Galton-Watson tree.

Consider first the case where $\mathbb{P}\{W = 0\} = 1$. In this case, explosion is equivalent to the event that the Galton-Watson tree is infinite, i.e., the survival of the Galton-Watson process. In that case, if $\mathbb{E}\{Z\} \le 1$, there is no survival, and if $\mathbb{E}\{Z\} > 1$, there is a positive probability of survival [4]. From now on we will assume that $\mathbb{P}\{W = 0\} < 1$ and assume that the Galton-Watson process is supercritical.

In the case of a supercritical Galton-Watson process, under the assumption $\mathbb{E}\{Z\} < \infty$, the results of Hammersley [21], Kingman [26], and Biggins [7] show the existence of a constant γ such that conditional on the non-extinction of the process, M_n/n tends to γ almost surely. This shows that the random variables M_n, conditional on survival, behave linearly in n, i.e., $M_n = \gamma n + o(n)$. One consequence of the Hammersley-Kingman-Biggins theorem is that if $\gamma > 0$, then explosion never happens. Now define

$$H := \mathbb{E}\{Z\}\mathbb{P}\{W = 0\}.$$

It can be shown that $\gamma = 0$ if and only if $H \ge 1$. We consider in fact three cases: $H < 1$, $H > 1$ and $H = 1$.

• Case I. $\quad H < 1$.

Here, as stated above, explosion occurs with probability zero. This can be seen more simply as follows: fix an $\epsilon > 0$ such that $\mathbb{P}\{W > \epsilon\} < (\mathbb{E}\{Z\})^{-1}$ and mark all edges with weight larger than ϵ. Then each component in the forest of marked edges is a subcritical Galton-Watson tree, and hence has finite size almost surely. Thus any infinite path must contain an infinite number of unmarked edges, and hence cannot be an exploding path.

• Case II. $\quad H > 1$.

In this case, explosion happens with probability one. To see this, take a sub-Galton-Watson tree by keeping only children for which $W = 0$. This tree is supercritical and thus survives with some positive probability ρ. It follows that with positive probability, there is an infinite path of length zero. Since, conditional on survival, explosion is a 0-1 event (for a proof see later in this introduction), we infer that it happens with probability one. A theorem of Dekking and

Host [15] ensures the existence of an almost surely finite random variable M such that M_n converges a.s. to M. Under the extra condition $\mathbb{E}\, Z^2 < \infty$, they determine stronger results on the limit distribution M.

• Case III. $\quad H = 1.$

This threshold case is the most intriguing—it was already considered in an earlier pioneering work of Bramson [10], and the work of Dekking and Host [15]. In this case, the occurrence of explosion is a delicately balanced event that depends upon the behavior of the distribution of W near the origin and on the distribution of Z.

Bramson's main theorem is the following result on the behaviour of M_n under the assumption that there exists a $\delta > 0$ such that $\mathbb{E}\{Z^{2+\delta}\} < \infty$. For any fixed λ, define $\sigma_{\lambda,n} = p + (1-p)e^{-\lambda^n}$ where $p = \mathbb{P}\{W = 0\} < 1$. Then explosion happens if and only if there exists some $\lambda > 1$ such that $\sum_{n=1}^{\infty} F_W^{-1}(\sigma_{\lambda,n}) < \infty$. In the case of no explosion, and conditional on the survival of the branching process, the following convergence result on the asymptotic of M_n holds. Almost surely, we have

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{s(n)} F_W^{-1}(\sigma_{2,k})} = 1, \tag{1}$$

where $s(n) = \lceil \log\log n / \log 2 \rceil$. We refer to [15] for a generalization of Bramson's theorem to the case of $\mathbb{E}\{Z^2\} < \infty$, under some extra mild conditions.

Following Bramson [10], we first transform the tree T_Z into a new tree T' as follows. The roots are identical. First consider the sub-Galton-Watson tree rooted at the root of T_Z consisting only of children (edges) that have zero weight. This subtree is critical. For any distribution of Z satisfying the threshold condition, note that the size S of the sub-Galton-Watson tree is a random variable $S \geq 1$ with $\mathbb{E}\{S\} = \infty$. In some cases, we know more—for example, when $\mathrm{Var}\{Z\} = \sigma^2 \in (0,\infty)$, then $\mathbb{P}\{S \geq k\} \sim \sqrt{2/\pi\sigma^2 k}$ as $k \to \infty$ (see, e.g., the book of Kolchin [27]). All of the nodes in S are mapped to the root of the new tree T'. The children of that root in T' are all the children of the mapped nodes in T_Z that did not have $W = 0$.

Let X_i be the number of vertices of degree i in the sub-Galton-Watson tree. The number of children of the root of T_Z is distributed as

$$\zeta = \sum_{i=0}^{\infty} \sum_{j=1}^{X_i} \zeta_{i,j},$$

where $\zeta_{i,1}, \zeta_{i,2}, \ldots$ are i.i.d. random variables having distribution of a random variable ζ_i. In addition the distribution of ζ_i is given by

$$\mathbb{P}\{\zeta_i = k\} = c_i \binom{k+i}{i} (1 - \mathbb{P}\{W=0\})^k \, \mathbb{P}\{W=0\}^i \, \mathbb{P}\{Z = k+i\},$$

where c_i is a normalizing constant. Note that $\sum_{i\geq 0} X_i = S$.

For each child of the root in T', repeat the above collapsing procedure. It is easily seen that T' itself is a Galton-Watson tree with offspring distribution ζ. The moment generating function $G_\zeta(s)$ of ζ is easily seen to satisfy the functional equation

$$G_\zeta(s) = G_Z\Big((1 - \mathbb{P}\{W=0\})s + \mathbb{P}\{W=0\} G_\zeta(s) \Big). \tag{2}$$

Furthermore, the displacement distribution is W conditional on $W > 0$. Finally, one can verify that $\mathbb{E}\{\zeta\} = \infty$. More importantly, explosion occurs in T_Z if and only if explosion happens in T'. We have thus reduced the explosion question to one for a new tree in which the expected number of children is infinite, and in which W does not have an atom at zero.

Observe that the transformation described in CASE III is valid whenever W has an atom at the origin. In particular, this construction can also be used to eliminate an atom at the origin when $\mathbb{P}\{W = 0\} > 0$ and $\mathbb{E}\{Z\} = \infty$. In this case, we still have $\mathbb{E}\{\zeta\} = \infty$.

It follows from the above discussion that in the study of the the event of explosion, we need to consider only the (most interesting) case where

$$\mathbb{E}\{Z\} = \infty,\ \mathbb{P}\{W = 0\} = 0.$$

All our results below are concerned only with this case.

A simple necessary condition for explosion. There is a rather obvious necessary condition for explosion. Let Y_i be the minimum weight edge at level i in the tree. Then the sum of weights along any infinite path is certainly at least $\sum_{i=1}^{\infty} Y_i$. We say that a fixed weighted tree is *min-summable* if this sum is bounded; if a tree is not min-summable, it cannot have an exploding path.

For any fixed, infinite, rooted tree T, and distribution W on the nonnegative reals, let T^W denote a random weighted tree obtained by weighting each edge with an independent copy of W. For a fixed tree T and weight distribution W, it follows easily from Kolmogorov's 0-1 law that explosion and min-summability of T^W are both 0-1 events. Thus, we make the following definitions.

Definition 1.1. For any infinite rooted tree T,

(i) let $\mathcal{W}_{\mathrm{EX}}(T)$ be the set of weight distributions so that T^W contains an exploding path almost surely, and

(ii) let $\mathcal{W}_{\mathrm{MS}}(T)$ be the set of weight distributions so that T^W is min-summable almost surely.

In this new notation, the observation above is simply that $\mathcal{W}_{\mathrm{EX}}(T) \subseteq \mathcal{W}_{\mathrm{MS}}(T)$, for any tree T. Unsurprisingly, in general $\mathcal{W}_{\mathrm{EX}}(T)$ may be strictly contained within $\mathcal{W}_{\mathrm{MS}}(T)$. For example, consider an infinite binary tree T and a uniform weight distribution W on $[0, 1]$. Except with probability at most $\exp(-2^{i/2})$ the minimum of 2^i copies of W is at most $2^{-i/2}$. Thus, with positive probability $\sum_{i\geq 1} Y_i \leq \sum_{i\geq 1} 2^{-i/2} < 3$, and so $W \in \mathcal{W}_{\mathrm{MS}}(Z)$. On the other hand, we may easily prove that $W \notin \mathcal{W}_{\mathrm{EX}}(Z)$, i.e., that the probability that there exists an exploding path is zero. To see this, consider the event A_i that there exists a path from the root to level i of weight less than $i/128$. The existence of an exploding path certainly implies that for all sufficiently large i, A_i occurs. We now observe that $\mathbb{P}\{A_i\} \leq 2^{-i}$. Indeed, the event A_i implies that there is a path from the root to level i at least half of whose edges have weight less than $\frac{1}{64}$. Since there are only 2^i paths to level i and at most 2^i ways to choose a subset of the edges of a fixed path, and since for each path and each fixed subset of at least $\frac{i}{2}$ edges, the probability that all these edges have weight less than $\frac{1}{64}$ is at most 8^{-i}, the bound easily follows. The same proof shows that for the exponential distribution E, no explosion can happen (however, $E \in \mathcal{W}_{\mathrm{MS}}(T)$; this follows from Example (iv) of Section 4).

Main results. It may appear that, aside from some trivial cases, $\mathcal{W}_{\text{MS}}(T)$ should always strictly contain $\mathcal{W}_{\text{EX}}(T)$. However, somewhat counter-intuitively, this is not the case; there are examples of trees with generation sizes growing very fast (double exponentially) for which $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$. Consider for example the tree T defined as follows: all nodes of generation n have 2^{2^n} children. In this case, for a given weight distribution W, the distribution of the sum of minimum weights of levels is

$$\sum_{n\geq 1} \min_{1\leq i\leq 2^{(2^n-1)}} W_n^i,$$

where each W_n^i is an independent copy of W. Also the path constructed by the simple greedy algorithm which, starting from root, adds at each step the lowest weight edge from the current node to one of its children, has total weight distributed as

$$\sum_{n\geq 1} \min_{1\leq i\leq 2^{2^{(n-1)}}} W_n^i.$$

The property of these sums being finite almost surely is clearly equivalent, so that $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$. Our main result is that this phenomenon is in fact quite general in trees obtained by a Galton-Watson process with a heavy tailed offspring distribution. We call the distribution Z *plump* if for some positive constant ϵ the inequality

$$\mathbb{P}\{Z \geq m^{1+\epsilon}\} \geq \frac{1}{m} \tag{3}$$

holds for all m sufficiently large. Equivalently, Z is plump if its distribution function F_Z satisfies $F_Z^{-1}(1-1/m) \geq m^{1+\epsilon}$ for m sufficiently large. We remark that $\mathbb{E}\, Z = \infty$ for any plump Z.

Equivalence Theorem. *Let Z be a plump distribution. Let T be a random Galton-Watson tree with offspring distribution Z, but conditioned on survival. Then*

$$\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T) \qquad \textit{with probability } 1.$$

We now state a second form of the equivalence theorem. For this, we must extend the definition of $\mathcal{W}_{\text{EX}}$ and $\mathcal{W}_{\text{MS}}$ to Galton-Watson offspring distributions. Let Z be an offspring distribution and W a weight distribution. We have

Claim. *For a given offspring distribution Z and weight distribution W, and conditioning on survival of the Galton-Watson process, explosion and min-summability are* 0-1 *events.*

Proof. Let $(W_i)_{i=1}^\infty$ be a sequence of independent copies of W, let $(S_i)_{i=1}^\infty$ be a random walk with jump distribution given by $Z-1$, and let $(X_i)_{i=1}^\infty$ be the increments. In the usual way, this random walk can be thought of as representing (in breadth-first fashion) a sequence of one or more Galton-Watson trees, with $X_i + 1$ giving the number of children at step i and W_i the weight of the i-th edge. Since $\mathbb{E}\, Z > 1$, one of these trees T' will be infinite with probability 1, and this tree is exactly a Galton-Watson tree conditioned on survival. The sequence $((X_i, W_i))_{i=1}^\infty$ clearly encodes all the information about T', and the two events under consideration are tail events with respect to this sequence; thus Kolmogorov's 0-1 law applies. The same argument holds for min-summability. □

We can thus define $\mathcal{W}_{\text{EX}}(Z)$ and $\mathcal{W}_{\text{MS}}(Z)$ for an offspring distribution Z as follows:

$$\mathcal{W}_{\text{EX}}(Z) := \Big\{ W \mid W \in \mathcal{W}_{\text{EX}}(T_Z) \text{ almost surely conditioned on survival} \Big\},$$

and

$$\mathcal{W}_{\text{MS}}(Z) := \Big\{ W \mid W \in \mathcal{W}_{\text{MS}}(T_Z) \text{ almost surely conditioned on survival} \Big\}.$$

The alternative (though slightly weaker) formulation of the Equivalence Theorem can now be stated as follows:

Equivalence Theorem – Second Version. *For a plump distribution Z,*

$$\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z).$$

Min-summability is clearly a simpler kind of condition than explosion; in particular, it depends only on the generation sizes Z_n rather than the full structure of the tree T_Z. Indeed, the Equivalence Theorem becomes more interesting if one observes that it is possible to derive the following quite explicit necessary and sufficient condition for min-summability.

Theorem 1.1. *Given a plump offspring distribution Z, let $m_0 > 1$ be large enough such that the condition (3) holds for all $m \geq m_0$. Define the function $h : \mathbb{N} \to \mathbb{R}^+$ as follows:*

$$h(0) = m_0 \qquad \textit{and} \qquad h(n+1) = F_Z^{-1}(1 - 1/h(n)) \quad \textit{for all } n \geq 1. \tag{4}$$

Then for any weight distribution W, $W \in \mathcal{W}_{\text{MS}}(Z)$, and hence also $W \in \mathcal{W}_{\text{EX}}(Z)$, if and only if

$$\sum_n F_W^{-1}(h(n)^{-1}) < \infty.$$

Given the Equivalence Theorem above, one may wonder if there is a way to weaken the condition given in (3) such that the theorem still remains valid. We show that this condition is to some extent the best we can ask for. More precisely, we prove

Sharpness of Condition 3. Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function satisfying

$$g(m) = m^{1+o(1)}.$$

Then there is an offspring distribution Z satisfying $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$ for all $m \in \mathbb{N}$, but for which $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$.

So far our results concerned the appearance of the event of explosion; however, it is also natural to ask how fast M_n tends to infinity in the case there is a.s. no exploding path. Although there is no reason to expect a convergence theorem in the case of no explosion for general plump distributions in the absence of any smoothness condition on the tails of Z, we show that a stronger plumpness property allows to obtain a precise information on the rate of convergence to infinity of M_n. To explain this, note that the plumpness assumption on Z is equivalent to $1 - F_Z(k) \geq k^{-\eta}$ for $\eta = \frac{1}{1+\epsilon}$ and for all k sufficiently large. Consider now the stronger smoothness condition

$$1 - F_Z(k) = k^{-\eta} \ell(k), \tag{5}$$

where ℓ is any continuous and bounded function which is nonzero at infinity.

Limit Theorem under Condition 5. *Let Z satisfy the smoothness condition, and let W be any weight distribution with $W \notin \mathcal{W}_{\text{EX}}(Z)$. Then a.s. conditional on survival,*

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{n} F_W^{-1}\Big(\exp\big(-(1+\epsilon)^k\big)\Big)} = 1$$

for all $\epsilon > 0$.

Applying a Tauberian theorem (see Section 7 for more details), we find that Condition 5 is equivalent to the condition

$$K_Z(s) := 1 - G_Z(1-s) \sim a\, s^{\eta} \ell(\frac{1}{s})$$

near $s = 0$ for some $a > 0$; recall G_Z is the moment generating function of Z. Going back to Case III of the finite mean case, and the transformation described there, we observe that the use of the functional Equation (2) allows to translate the smoothness condition above, imposed on the modified offspring distribution ζ of infinite mean (obtained after the transformation), to a smoothness condition on Z, the original distribution of finite mean. In particular,

$$K_\zeta(s) = 1 - G_\zeta(1-s) \sim a s^{1/(1+\epsilon)}\big(1 + O(s^\beta)\big) \qquad \text{for } s \text{ near zero,}$$

for some $a, \epsilon, \beta > 0$ is equivalent to a condition of the form

$$K_Z(s) \sim \mathbb{E}\{Z\}\, s - c\, s^{1+\epsilon}(1 + O(s^\delta)) \qquad \text{for } s \text{ near zero,} \tag{6}$$

for some $c, \delta > 0$. We note that Condition 6 assumes some regularity on the tails of Z but the variance could be infinite; thus, the above result can be regarded as a strengthening of Bramson's theorem [10].

Further related work. The literature on explosion is partially surveyed by Vatutin and Zubkov [38]. The early work deals with exponentially distributed weights: in this case, there is no explosion almost surely if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n \sum_{r=0}^{n} \mathbb{P}\{Z > r\}} < \infty$$

(see [22, V. 6], [30, 17]). This condition cannot be simplified; Grey [20] showed that there does not exist any fixed function $\psi \geq 0$ such that explosion would be equivalent to $\mathbb{E}\{\psi(Z)\} = \infty$.

Some general properties of the event of explosion were obtained in [33] by considering the generating functions of the number of particles born before time t, parametrized by t, and looking at the nonlinear integral equation satisfied by these generating functions. By using this analytic approach, and under some smoothness conditions on the distribution function F_W of the displacement W, Sevast'yanov [33, 34], Gel'fond [19], and Vatutin [35, 36] obtain necessary and sufficient conditions on the event of explosion. The result of Vatutin [36] can be stated as follows. Consider the case $\mathbb{P}\{W = 0\} = 0$ and suppose that zero is an accumulation point of W, i.e., the distribution function F_W of W satisfies $F_W(w) > 0$ for all $w > 0$. Assume the following regular variation style condition holds: there exists $\lambda \in (0, 1)$ such that

$$0 < \liminf_{t\downarrow 0} \frac{F_W^{-1}(\lambda t)}{F_W^{-1}(t)} \leq \limsup_{t\downarrow 0} \frac{F_W^{-1}(\lambda t)}{F_W^{-1}(t)} < 1. \tag{7}$$

Then explosion does not occur if and only if for all $\epsilon > 0$,

$$\int_0^{\epsilon} F_W^{-1}\Big(\frac{s}{K_Z(s)}\Big)\frac{ds}{s} = \infty. \tag{8}$$

Condition (7) basically forces F_W to behave in a polynomial manner near the origin. Indeed, if $F_W(w) \sim w^{\alpha}$ for some $\alpha > 0$ as $w \downarrow 0$, then $F_W^{-1}(t) \sim t^{1/\alpha}$ as $t \downarrow 0$, and so (7) holds. The exponential law corresponds to $\alpha = 1$, for example. The criterion given by (8) was earlier proved to be necessary and sufficient for non-explosion by Sevast'yanov [33, 34] and Gel'fond [19] under the slightly more restrictive condition that $F_W(w)/w^{\alpha} \in [a, b]$ for all w, where $0 < a \le b < \infty$ and $\alpha \ge 0$. As soon as we leave that polynomial oasis, Vatutin's condition is violated. Examples include $F_W(w) \sim \exp(-1/w^{\alpha})$ and $F_W(w) \sim 1/\log^{\alpha}(1/w)$ for $\alpha > 0$.

A quite general sufficient (but not necessary) condition without any explicit regularity assumption on W was proved by Vatutin [37] for explosion in non-homogenous branching random walks. In the homogenous case, the result states that if there exists a sequence of nonnegative reals $(y_n)_{n\in\mathbb{N}}$ such that $\lim_n y_n = 0$ and

$$\sum_{n=1}^{\infty} F_W^{-1}\big(y_n/K_{Z_n}(y_n)\big) < \infty,$$

then explosion occurs. This result is close in spirit to our equivalence theorem, but we stress that the results are distinct—we see no way in which one may be deduced from the other.

More precise information on the behaviour and convergence to infinity of M_n can be obtained in the finite mean case and under extra conditions. Recall that in the finite mean case, $M_n = \gamma n + o(n)$ for some $\gamma \ge 0$. McDiarmid showed in [28] that $M_n - \gamma n = O(\log n)$ under the condition $\mathbb{E}\{Z^2\} < \infty$. Recently, Hu and Shi [23] proved that if the displacements are bounded and $\mathbb{E}\{Z^{1+\epsilon}\} < \infty$ for any $\epsilon > 0$, then conditional on survival, $(M_n - \gamma n)/\log n$ converges in probability but, interestingly, not almost surely. (We note in passing that this work and the recent work of Aïdekon-Shi [3] provide Seneta-Heyde norming results [9] in the boundary case.) Under the extra assumption that Z is bounded, Addario-Berry and Reed [1] calculate $\mathbb{E}\{M_n\}$ to within $O(1)$ and prove exponential tail bounds for $\mathbb{P}\{|M_n - \mathbb{E}\{M_n\}| > x\}$. Extending these results, Aïdekon [2] proves the convergence of M_n centered around its median for a large class of branching random walks. For tightness results in general, under some extra assumptions on the decay of the tail distribution or weight distribution, see Bachmann [5] and Bramson-Zeitouni [11, 12].

Organization of the paper. Section 2 will concern some preliminaries, mostly involving what we call the *speed* of an offspring distribution. In Section 3, we prove the Equivalence Theorem. The proof is somewhat algorithmic in nature, and shows that a certain (infinite) algorithm will always find an exploding path under the given conditions. A second proof of the Equivalence Theorem is sketched in Section 5. In Section 4, we prove Theorem 1.1, and give some examples calculating the condition for specific cases. In Section 6 we provide a generic counterexample that shows that the equivalence does not hold if we weaken the conditions in any substantial way, proving the sharpness of Condition 3. Finally, in Section 7 we prove the limit theorem under Condition 5.

2 Preliminaries

In this section we present some definitions and results needed for the proof of the Equivalence Theorem. That theorem (in its second form) is concerned with the equivalence of $\mathcal{W}_{\text{MS}}(Z)$ and $\mathcal{W}_{\text{EX}}(Z)$ for certain offspring distributions Z. Thus it will be important to have a good characterization of whether a weight distribution W belongs to $\mathcal{W}_{\text{MS}}(Z)$; in other words, whether $\sum_{n\geq 1}\min\{W_n^1,\ldots,W_n^{Z_n}\}$ is finite, each W_n^i being an independent copy of W. To do this we will introduce two notions. The first is the concept of the *speed* of a branching process, from which we will obtain an understanding of the growth of the generation sizes Z_n. The second is the concept of *summability with respect to an integer sequence*, which concerns the behaviour of sums of the form $\sum_{n\geq 1}\min\{W_n^1,\ldots,W_n^{\sigma_n}\}$ for a given integer sequence $(\sigma_n)_{n\in\mathbb{N}}$.

Speed of a Galton-Watson branching process. We introduce the concept immediately and then give a number of examples.

Definition 2.1. An increasing function $f:\mathbb{N}\to\mathbb{R}^+$, taking only strictly positive values, is called a *speed* of a Galton-Watson offspring distribution Z if there exist positive integers a and b such that with positive probability

$$Z_{n/a}\leq f(n)\leq Z_{bn}\qquad \text{for all } n\in\mathbb{N}.$$

(Here, we set $Z_x = Z_{\lfloor x\rfloor}$ for $x\in\mathbb{R}$.)

Note that there is a small issue of extinction here, and that is why we insist that f is strictly positive, otherwise $f(n)=0$ would be a speed for any distribution with $\mathbb{P}\{Z=0\}>0$.

Examples of speeds. Here we give examples of speeds for various distributions Z.

(i) If $\mathbb{E}\{Z\}\leq 1$ then almost surely $Z_n=0$ for all sufficiently large n, and so Z does not have a speed.

(ii) If $\mathbb{E}\{Z\}=m\in(1,\infty)$, then Doob's limit law states that the random variable $V_n=Z_n/m^n$ form a martingale sequence with $\mathbb{E}\,V_n\equiv 1$, and $V_n\to V$ almost surely, where V is a nonnegative random variable. Furthermore, in the case that Z is bounded the limit random variable V has mean 1 (and so in particular $\mathbb{P}\{V\geq 1\}>0$). From this we may easily verify that m^n is a speed of Z. Indeed Doob's limit law implies that the inequality $Z_n\leq(M+1)m^n$ holds for all n large enough, with probability at least $P(V\leq M)$. Taking M sufficiently large, this probability may be made arbitrarily close to 1. For the lower bound, one may consider a truncation Z' of Z such that $\mathbb{E}\{Z'\}\geq\sqrt{m}$. Since Z' is bounded, we deduce that in the truncated branching process associated with Z' there is a positive probability that $Z'_n\geq m^{n/2}/2$ for all sufficiently large n. Since there is a natural coupling such that $Z_n\geq Z'_n$ for all n, this completes our proof that m^n is a speed of Z.

(iii) If Z is defined by $\mathbb{P}\{Z\geq m+1\}=m^{-\beta}$ for each $m\geq 1$, where $\beta\in(0,1)$, then Z is plump (one may take $\epsilon=\beta^{-1}-1$ in Condition 3) and the double exponential function $f(n)=2^{(\beta^{-1})^n}$ is a speed of Z. Heuristically, this follows from the fact that conditioned on the value of Z_n one would expect Z_{n+1} to be of the order $Z_n^{\beta^{-1}}$. A formal proof follows from Theorem 2.3 together with the observation that the function h appearing in that

theorem is equivalent to f as a speed (i.e., there exist $a', b' \in \mathbb{N}$ such that the inequalities $f(\lfloor n/a' \rfloor) \le h(n) \le f(b'n)$ hold for all n). Indeed, as we will explain in Section 7, a much stronger statement holds in this case.

(iv) If Z is defined by $\mathbb{P}\{Z \ge m\} = 1/\log_2 m$ for each $m \ge 2$, then Z is plump. Applying Theorem 2.3 we find that the tower function $h(n)$ defined by $h(0) = 2$ and $h(n+1) = 2^{h(n)}$ for $n \ge 0$ is a speed of Z.

Summable weight distributions with respect to an integer sequence. Let W be a random variable with nonnegative values. Let $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of positive integers and W_n^j be a family of independent copies of W for $n, j \in \mathbb{N}$. Define the sequence of minima

$$\Lambda_n := \min_{1\le j\le \sigma_n} W_n^j.$$

The random variable W is called *σ-summable* if there is a positive probability that $\sum_n \Lambda_n$ is finite.

Note that the event in the above definition is a 0-1 event. Thus, if W is σ-summable, then $\sum_n \min_{1\le j\le\sigma_n} W_n^j$ is finite with probability one. For a characterization of σ-summable weight distributions see Proposition 4.1. Examples are given at the end of Section 4.

We note that if W is σ-summable and τ-summable, then W is $\sigma \cup \tau$-summable, and if $\sigma_n \le \tau_n$ for all n, σ-summability implies τ-summability. We also have

Lemma 2.1. *Let σ be any increasing sequence, and let τ be defined by $\tau_n = \sigma_{\gamma n}$ for some constant γ, a positive integer. Then W is σ-summable iff it is τ-summable.*

Proof. Write $\sigma = \sigma^0 \cup \sigma^1 \cup \cdots \cup \sigma^{\gamma-1}$, where $\sigma^i := \{\sigma_{\gamma n+i} : n \in \mathbb{N}\}$. Since σ is increasing, if W is σ^i-summable and $i < j$, then W is σ^j-summable. So if W is $\tau = \sigma^0$-summable, then it is σ^i-summable for all $0 \le i \le \gamma - 1$, and thus σ-summable. The other direction follows trivially since $\tau \subseteq \sigma$. □

The following proposition relates the condition of the Equivalence Theorem to the notion of σ-summability under the presence of a speed function for the Galton-Watson distribution.

Proposition 2.2. *Let W be a weight distribution and Z an offspring distribution. Suppose that $f : \mathbb{N} \to \mathbb{R}^+$ is a speed for Z. Then $W \in \mathcal{W}_{\text{MS}}(Z)$ if and only if W is σ-summable for the sequence $\sigma = (f(n))_{n\in\mathbb{N}}$.*

Proof. Since f is a speed for Z, the event

$$R := \big\{Z_{n/a} \le f(n) \le Z_{bn} \text{ for all } n\big\}$$

occurs with positive probability. Let σ^a be the sequence given by $\sigma_n^a = f(an)$, and σ^b the sequence defined by $\sigma_n^b = f(\lfloor n/b \rfloor)$. Suppose W is σ-summable; then by Lemma 2.1, W is σ^b-summable. Whenever R occurs, $Z_n \ge \sigma_n^b$ for all n, and hence T_Z has the min-summability property almost surely. Thus, $W \in \mathcal{W}_{\text{MS}}(T_Z)$ with positive probability, and hence $W \in \mathcal{W}_{\text{MS}}(Z)$.

Conversely, if W is not σ-summable, then again by Lemma 2.1, it is not σ^a-summable. Thus, even when conditioning on survival, $W \notin \mathcal{W}_{\text{MS}}(T_Z)$ with positive probability, and hence $W \notin \mathcal{W}_{\text{MS}}(Z)$. □

Definition of a speed function for plump distributions Z. We are now in a position to partially explain the mysterious function h defined in (4), which recall was defined by

$$h(0) = m_0 \qquad \text{and} \qquad h(n+1) = F_Z^{-1}(1 - 1/h(n)).$$

It will turn out that this function defines a speed function for the offspring distribution Z in the sense of Definition 2.1.

Theorem 2.3. *If the offspring distribution Z is plump, then the function h is a speed of Z.*

Although it is possible to present a proof at this stage, to avoid redundancy we postpone it until Section 3.

It will actually be convenient in our proofs to consider a slight variation on h. Let $\alpha = (1+\epsilon)^{-1/2}$, and define f by

$$f(0) = \tilde{m}_0 \qquad \text{and} \qquad f(n+1) = F_Z^{-1}(1 - f(n)^{-\alpha}), \tag{9}$$

where $\tilde{m}_0$ is the least integer such that Condition 3 holds with $m_0 = \tilde{m}_0^\alpha$, and the following inequalities hold: $\tilde{m}_0^{1-\alpha} \geq 16(1-\alpha)^{-1} + 16$ and $\tilde{m}_0^{\alpha^{-1}-1} \geq 4^{\lceil(\alpha^{-1}-1)^{-1}\rceil+1}$.

The functions h and f are essentially equivalent as far as we are concerned. The following lemma demonstrates their equivalence as speeds.

Lemma 2.4. *For any plump distribution Z, h is a speed for Z if and only if f is.*

Proof. Since h is increasing, for some constant c we have $h(c) \geq \tilde{m}_0 = f(0)$. Inductively, we then have $f(n) \leq h(n+c)$ for all n. Since Z is plump, we have from the definition of f that

$$f(n+1) \geq f(n)^{\alpha(1+\epsilon)} = f(n)^{1/\alpha} \qquad \text{for any } n.$$

Thus,

$$f(n+2) = F_Z^{-1}(1 - f(n+1)^{-\alpha}) \geq F_Z^{-1}(1 - f(n)^{-1}).$$

It follows that if $f(n) \geq h(m)$, then $f(n+2) \geq h(m+1)$. So by induction, we have $f(2n) \geq h(n)$.

Considering the definition of a speed for Z, we see that if one is a speed, so is the other. ☐

In the following lemma, we state some direct consequences of Condition 3 (i.e. the assumption Z is plump), and the definition of f, that will be helpful later.

Lemma 2.5. *Let Z be a plump distribution and let $f(n)$ be defined as in (9).*

(i) For all n,

$$f(n+2) \geq F_Z^{-1}(1 - 1/f(n)). \tag{10}$$

(ii) $f(n+1) \geq 4^{n+1} f(n)$ for all $n \geq 0$. In particular, $f(n)^{1-\alpha} \geq 16n + 16$ for all $n \geq 1$, and for any positive r, $f(n) = \Omega(r^n)$.

(iii) For each $k \geq 2$ and for all n,

$$f(n + 2\lceil \log k / \log(1+\epsilon) \rceil) \geq f(n)^k. \tag{11}$$

Proof. Part (i) follows immediately from the proof of Lemma 2.4. To prove Part (ii) we begin by noting that the ratio $f(n+1)/f(n)$ is at least $f(n)^{\alpha^{-1}-1}$, as $\alpha(1+\epsilon)=\alpha^{-1}$. We therefore prove that $f(n)^{\alpha^{-1}-1} \geq 4^{n+1}$ for all n. Let $n_0 = \lceil (\alpha^{-1}-1)^{-1} \rceil$, and note that since $\tilde{m}_0^{\alpha^{-1}-1} \geq 4^{\lceil (\alpha^{-1}-1)^{-1} \rceil + 1}$, the inequality $f(n)^{\alpha^{-1}-1} \geq 4^{n+1}$ holds trivially for $n \leq n_0$. For $n > n_0$, the result follows easily by induction as

$$f(n)^{\alpha^{-1}-1} \geq (4^n f(n-1))^{\alpha^{-1}-1} = 4^{(\alpha^{-1}-1)n} f(n-1)^{\alpha^{-1}-1} \geq 4 f(n-1)^{\alpha^{-1}-1}.$$

To conclude the proof of Part (ii), we have to show $f(n)^{1-\alpha} \geq 16n + 16$ for all n. For $n \leq (1-\alpha)^{-1}$, we trivially have

$$f(n)^{1-\alpha} \geq f(0)^{1-\alpha} = \tilde{m}_0^{1-\alpha} \geq 16(1-\alpha)^{-1} + 16.$$

For $n \geq (1-\alpha)^{-1} + 1$, we have $f(n)^{1-\alpha}/f(n-1)^{1-\alpha} \geq 4$, and the result easily follows by induction.

To prove Part (iii), we note that

$$f(n+2) = F_Z^{-1}(1 - 1/f(n)) \geq f(n)^{1+\epsilon}.$$

An inductive argument now easily yields that

$$f(n+2\ell) \geq f(n)^{(1+\epsilon)^\ell}$$

for any n and ℓ. It follows that $f(2n) \geq m_0^{(1+\epsilon)^n}$. We conclude by setting $\ell = \lceil \log k / \log(1+\epsilon) \rceil$. ☐

3 Proof of the Equivalence Theorem

In this section we prove the Equivalence Theorem. We first prove it in the second (technically weaker) form and then describe how the first form may be deduced.

Let Z be a plump offspring distribution, and let ϵ and m_0 be such that Condition 3 holds for the triple Z, ϵ, and m_0. Fix an arbitrary $W \in \mathcal{W}_{\text{MS}}(Z)$. We shall prove that $W \in \mathcal{W}_{\text{EX}}(Z)$ (and the theorem will follow). We define an algorithm which selects a path in the tree in a very precise way; then using the properties of W, we prove that with positive probability this path is an exploding path. Since, conditioned on survival, the event that there is an exploding path is a 0-1 event, this is enough to prove the theorem.

The algorithm depends on a parameter α, defined in the previous section: $\alpha := (1+\epsilon)^{-\frac{1}{2}}$. The reason for this choice of exponent will be clarified later in the proof.

Algorithm FindPath:

Let x_0 be the root of the tree.

For $n = 0, 1, 2, \ldots,$

– Consider node x_n, which is the lowest node in the candidate exploding path we are constructing. Let Y_{n+1} denote the number of children of x_n.

– Order the children of x_n by how many children they in turn have, from largest to smallest. Let $X_{n+1} := \lceil (Y_{n+1})^{(1-\alpha)}/2 \rceil$. We define the *options* from x_n to be the first X_{n+1} children of x_n in the ordering.

– If $X_{n+1} = 0$, the algorithm terminates in failure. Otherwise, of the X_{n+1} choices, pick the option whose edge from x_n has the smallest weight, and set x_{n+1} to be this child.

The analysis of the algorithm, and the proof that it provides with positive probability an exploding path, will be based on the following assertion.

Claim. *There exists a positive integer a such that, with positive probability, $Z_n \leq f(an)$ and $Y_n \geq f(n)$ hold simultaneously for all $n \in \mathbb{N}$, where f is the function defined in Equation (9).*

Indeed, given this, we may deduce immediately that with positive probability $Z_{n/a} \leq f(n) \leq Z_n$ for all $n \in \mathbb{N}$, implying that $f(n)$ is a speed of Z. Furthermore, since X_n, the number of options of x_{n-1}, is defined by $X_n = \lceil Y_n^{(1-\alpha)}/2 \rceil$, there is a positive probability that $X_n \geq f(n-\gamma)$ for all $n \in \mathbb{N}$, where $\gamma = 2\lceil \log{(1-\alpha)^{-1}}/\log(1+\epsilon) \rceil + 1$ (this follows from Lemma 2.5 (iii)).

We now observe that conditional on the inequality $X_n \geq f(n-\gamma)$ holding for all $n \in \mathbb{N}$, the path $x_0, x_1, x_2, \dots$ is an exploding path almost surely. The distribution of the sum of weights along the path $x_0, x_1, x_2, \dots$, dependent on $X_1, X_2, X_3, \dots$, is given by

$$\sum_{n \geq 1} \min \left\{ W_n^1, \dots, W_n^{X_n} \right\},$$

where the W_n^j are i.i.d with distribution W. Thus, conditional on the event that $X_n \geq f(n-\gamma)$ for all $n \in \mathbb{N}$, this sum is stochastically smaller than $\sum_{n \geq 1} \min \left\{ W_n^1, \dots, W_n^{f(n-\gamma)} \right\}$. Moreover, Lemma 2.1 implies that W is σ-summable for the sequence $\sigma = (f(n))_{n \in \mathbb{N}}$, and since the contribution of any finite number of terms is finite, W is also σ-summable for the sequence $\sigma = (f(n-\gamma))_{n \in \mathbb{N}}$. This proves that $x_0, x_1, x_2, \dots$ is an exploding path almost surely.

So it remains to prove Claim 3, which we will do for the choice $a = 3 + 2\lceil \log 2/\log(1+\epsilon) \rceil$.

Define the two families of events $\{A_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$ by

$$A_n := \Big\{ Y_n < f(n) \Big\}, \qquad B_n := \Big\{ Z_n > f(an) \Big\}.$$

We are led to prove that there is a positive probability that none of the events A_n or B_n occur. Let $C = A_1^c \cap B_1^c$. The definition of f implies that Z assigns a positive probability to the range $[f(1), f(a)]$, so that $\mathbb{P}\{C\} > 0$. We will show below that

$$\mathbb{P}\{A_2 \mid C\} \leq 1/16 \quad \text{and} \quad \mathbb{P}\{A_{n+1} \mid A_n^c\} \leq 4^{-n-1} \quad \text{for } n \geq 2; \tag{12}$$

$$\mathbb{P}\{B_2 \mid C\} \leq 1/16 \quad \text{and} \quad \mathbb{P}\{B_{n+1} \mid B_n^c\} \leq 4^{-n-1} \quad \text{for } n \geq 2. \tag{13}$$

Assuming the above inequalities, we infer that

$$\begin{aligned}\mathbb{P}\Big\{ C \cap \bigcap_{n\geq 1} A_{n+1}^c \Big\} &= \mathbb{P}\{C\} \prod_{n\geq 1} \mathbb{P}\Big\{ A_{n+1}^c \mid A_n^c, A_{n-1}^c, \ldots, A_2^c, C \Big\} \\ &= \mathbb{P}\{C\}\, \mathbb{P}\big\{ A_2^c \mid C \big\} \prod_{n\geq 2} \mathbb{P}\big\{ A_{n+1}^c \mid A_n^c \big\} \\ &\qquad \text{(Since the sequence } Y_1, Y_2, Y_3, \ldots \text{ is Markovian)} \\ &\geq \big(1 - \sum_{n\geq 1} 4^{-n-1}\big)\mathbb{P}\{C\}.\end{aligned}$$

In the same way, we obtain $\mathbb{P}\Big\{ C \cap \bigcap_{n\geq 1} B_{n+1}^c \Big\} \geq \big(1 - \sum_{n\geq 1} 4^{-n-1}\big)\mathbb{P}\{C\}$. Since both the events $C \cap \bigcap_{n\geq 1} A_{n+1}^c$ and $C \cap \bigcap_{n\geq 1} B_{n+1}^c$ are contained in C, we conclude that with positive probability none of the events A_n and B_n occur, finishing the proof of the claim.

All that remains is to prove inequalities (12) and (13). We first prove the bound on $\mathbb{P}\big\{ A_{n+1} \,|\, A_n^c \big\}$ (it will be seen that the bound on $\mathbb{P}\big\{ A_2 \,|\, C \big\}$ follows by the same proof). Call a child of x_n *good* if it has at least $f(n+1)$ children, and write G_n for the number of good children of x_n. We note that given Y_n, the distribution of G_n is $\mathrm{Bin}(Y_n, p)$ where p, the probability that a given child is good, is at least $1 - F_Z(f(n+1)) = f(n)^{-\alpha}$. By the way the algorithm chooses the vertex x_{n+1}, we also note that A_{n+1} can occur only if $G_n < Y_n^{1-\alpha}/2$. Thus, conditional on $Y_n \geq f(n)$, if A_{n+1} occurs then

$$G_n \,<\, Y_n^{1-\alpha}/2 \,\leq\, Y_n f(n)^{-\alpha}/2 \,\leq\, \mathbb{E}\{G_n\}/2.$$

Hence

$$\begin{aligned}\mathbb{P}\big\{A_{n+1} \mid A_n^c\big\} &\leq \mathbb{P}\Big\{ G_n \leq \frac{Y_n^{1-\alpha}}{2} \,\Big|\, Y_n \geq f(n) \Big\} \\ &\leq \exp\Big(\frac{-f(n)^{1-\alpha}}{8}\Big) \\ &\leq \frac{1}{4^{n+1}}. && \text{(By Lemma 2.5 (ii))}\end{aligned}$$

We now prove $\mathbb{P}\{B_{n+1} \mid B_n^c\} \leq 4^{-(n+1)}$ (the proof bounding $\mathbb{P}\{B_2 \mid C\}$ being identical). Note that by Lemma 2.5 (iii),

$$f(an+a) \geq f(an) f(an+3).$$

Thus in order for the event $Z_{n+1} \geq f(an+a)$ to occur, conditional on $Z_n \leq f(an)$, there must be some node in generation n having at least $f(an+3)$ children. Taking $Z(i)$ to be an independent copy of Z for each i, the probability of this is bounded as follows:

$$\begin{aligned}\mathbb{P}\big\{\max\{Z(1), \ldots, Z(f(an))\} > f(an+3)\big\} &\leq f(an)\,\mathbb{P}\big\{Z > f(an+3)\big\} \\ &\leq f(an)\big(1 - F_Z(f(an+3)\big) \\ &\leq f(an) f(an+1)^{-1} \\ &\qquad \text{(By Lemma 2.5 (i))} \\ &\leq \frac{1}{4^{n+1}}. \text{ (By Lemma 2.5 (ii))}\end{aligned}$$

The proof of the Equivalence Theorem (in its second form) is complete. Note that in the process, we have also proved that f is a speed of Z; thus by Lemma 2.4, Theorem 2.3 also follows.

First form of the Equivalence Theorem. One might hope that the first form of the Equivalence Theorem could be deduced from the second by some very simple reasoning, perhaps considering for each weight distribution W the set of trees T for which $\mathcal{W}_{\text{EX}}(T) \neq \mathcal{W}_{\text{MS}}(T)$. However, the fact that there are uncountably many possible weight distributions seems to be problematic for such a direct approach.

Taking T to be a random Galton-Watson tree with offspring distribution Z conditioned to survive, we will prove that the following chain of containments holds almost surely:

$$\mathcal{W}_{\text{MS}}(T) \subseteq \mathcal{W}_{\text{MS}}(Z) \subseteq \mathcal{W}_{\text{EX}}(T).$$

From this the Equivalence Theorem in its first form immediately follows.

That the first inclusion holds almost surely follows from the fact that the rate of growth of generation sizes of T may almost surely be bounded in terms of the speed f of Z. Specifically, taking $a = 3 + 2\lceil \log 2/\log(1+\epsilon)\rceil$ as in Claim 3, we will show that almost surely there exists a constant c such that $Z_n \leq f(an + c)$ for all n. For $z \in \mathbb{N}$, let $r(z)$ denote the greatest r for which $z \geq f(r)$. If no bound of the form $Z_n \leq f(an + c)$ holds, then there must be infinitely many n for which $r(Z_{n+1}) > r(Z_n) + a$. However, our proof of (13) demonstrates that the probability that $Z_{n+1} \geq f(r + a)$ given that $Z_n \leq f(r)$ is at most 4^{-r}. Since f is a speed of Z, the sequence of probabilities $4^{-r(Z_n)}$ is summable almost surely, and so this event has probability zero.

That the second inclusion holds almost surely follows from the fact that we may apply the above algorithmic approach to finding an exploding path to any rooted subtree of T which survives. For a node v, let T_v denote the subtree of its descendants. Denote by $s(n)$ the number of nodes of generation n for which T_v is infinite. As T is conditioned on survival, the function $s(n)$ is unbounded almost surely [4, Ch. 10–12]. Let now $W \in \mathcal{W}_{\text{MS}}(Z)$. The above algorithm, applied independently to each node of generation n for which T_v is infinite, has positive probability $p > 0$ of producing an exploding path in each. Thus the probability of no exploding path is at most $(1-p)^s$ for all s, and so is 0.

The set of weights of infinite rooted paths. The following theorem characterizes the set of all possible values the weights of infinite rooted paths can take conditioned on the survival of the Galton-Watson tree. Note that the theorem is valid in general and does not require the plumpness condition.

Theorem 3.1. *Let Z be an offspring distribution and W a nonnegative weight distribution which is not a.s. zero. Then almost surely conditioned on survival the set of weights of infinite rooted paths is $[A, \infty]$ where A is the infimum weight of infinite rooted paths.*

Proof. By applying the transformation discussed in the introduction if necessary, we may assume that W has no atom at zero. Note that clearly the transformation does not change the weights of infinite rooted paths.

The theorem is clearly true if $W \notin \mathcal{W}_{\text{EX}}(Z)$ since in this case, conditioned on survival, all infinite rooted paths have infinite weight. So in the following we assume $W \in \mathcal{W}_{\text{EX}}(Z)$.

By a straightforward compactness argument, it suffices to show that for any $\epsilon' > 0$, there exists (almost surely) an infinite path with weight in $[a, a+\epsilon']$, for all $a \geq A$.

Let $\epsilon \leq \epsilon'/3$ be such that $\mathbb{P}\{W \in (\epsilon, 2\epsilon)\} > 0$; such an ϵ must exist since $W \in \mathcal{W}_{\text{EX}}(Z)$ and W has no atom at zero. Define the *path-weight* $\text{pw}(v)$ of a node v to be the sum of the edge weights on the path from v to the root. Now let

$$S_i = \{v \in T \mid \text{pw}(v) \in [i\epsilon, (i+1)\epsilon)\}.$$

The choice of ϵ is such that if $v \in S_i$, then for any given child w of v, $w \in S_{i+1} \cup S_{i+2}$ with a constant positive probability.

Since explosion occurs, there is some least integer ℓ such that S_ℓ is infinite (indeed, ℓ is so that $A \in [\ell\epsilon, (\ell+1)\epsilon)$). We may explore $S_0, S_1, \ldots$ in turn, each time uncovering all of S_i, as well as all children of nodes in S_i. In the process of exploring S_ℓ, each node we explore whose parent is in S_ℓ will have a constant positive probability of being in $S_{\ell+1} \cup S_{\ell+2}$; thus a.s. at least one of $S_{\ell+1}$ and $S_{\ell+2}$ is infinite too. Continuing inductively, we find that a.s. for any integer $j \geq \ell$, one of the sets S_j or S_{j+1} should be infinite. Obviously each infinite S_j contains an infinite rooted path of weight in the interval $[j\epsilon, (j+1)\epsilon]$. The result now follows by taking j with $a \in [j\epsilon, (j+1)\epsilon)$: since one of the two sets S_{j+1}, S_{j+2} is infinite, we infer the existence of an infinite path with length in the interval $[a, a+3\epsilon] \subseteq [a, a+\epsilon']$. □

4 Equivalent conditions for min-summability

In the previous section, we proved an equivalence theorem between explosion and min-summability for branching processes with plump offspring distributions. Though, the existence of such a result is certainly nice in its own, one may wonder if the property of min-summability is in any sense substantially simpler than that of explosion. The aim of this section is to answer this question in the affirmative by proving Theorem 1.1, which provides a necessary and sufficient condition for min-summability that involves a calculation based only on the distributions. We then provide some examples at the end of this section.

Let W be a random variable taking values in $[0, \infty)$ and let $\sigma = (\sigma_i)_{i \geq 0}$ be a sequence of positive integers. Then we have

Proposition 4.1. *The nonnegative random variable W is σ-summable if and only if the following two conditions are satisfied:*

$$\text{(i)} \qquad \sum_n \big(\mathbb{P}\{W > 1\}\big)^{\sigma_n} < \infty, \qquad \text{and}$$

$$\text{(ii)} \qquad \sum_n \int_0^1 \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} \, dt < \infty.$$

Proof. As in Section 2, let W_n^j be an independent copy of W for each $n, j \in \mathbb{N}$ and let

$$\Lambda_n := \min_{1 \leq j \leq \sigma_n} W_n^j.$$

Clearly, Λ_n is a sequence of nonnegative and independent random variables. By Kolmogorov's three-series theorem (see, e.g., Kallenberg [24] or Petrov [29]), we have $\sum_n \Lambda_n < \infty$ almost

surely if and only if

$$\sum_n \mathbb{P}\{\Lambda_n > 1\} < \infty,$$

$$\sum_n \mathbb{E}\left\{\Lambda_n \ \mathbf{1}_{[\Lambda_n \le 1]}\right\} < \infty,$$

$$\text{and} \qquad \sum_n \operatorname{Var}\left\{\Lambda_n \ \mathbf{1}_{[\Lambda_n \le 1]}\right\} < \infty.$$

Since W is nonnegative, random variables $\Lambda_n \ \mathbf{1}_{[\Lambda_n \le 1]}$ take value in $[0,1]$, and so the third condition follows from the second one. Now, $\mathbb{P}\{\Lambda_n > 1\} = (\mathbb{P}\{W > 1\})^{\sigma_n}$, and $\mathbb{E}\big\{\Lambda_n \ \mathbf{1}_{[\Lambda_n \le 1]}\big\} = \left(\int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} \, dt\right) - \mathbb{P}\{\Lambda_n > 1\}$, thus proving the theorem. □

In the case of a random integer sequence given by the generation sizes, it is also possible to give a result analogous to Proposition 4.1 (whose proof is omitted).

Proposition 4.2. *Let $\{Z_n\}$ be a Galton-Watson process with an offspring distribution Z, satisfying $Z \ge 1$ almost surely. Let Λ_n be the minimum weight of the n-th generation. We have*

$$\mathbb{P}\Big\{\sum_n \Lambda_n < \infty\Big\} = 1$$

if and only if the following two conditions are satisfied

$$\textit{(i)} \qquad \mathbb{P}\Big\{ \sum_n \big(\mathbb{P}\{W > 1\}\big)^{Z_n} < \infty \Big\} = 1, \qquad \textit{and}$$

$$\textit{(ii)} \qquad \mathbb{P}\Big\{ \sum_n \int_0^1 \big(\mathbb{P}\{W > t\}\big)^{Z_n} \, dt < \infty \Big\} = 1.$$

Otherwise, $\mathbb{P}\big\{\sum_n \Lambda_n < \infty\big\} = 0$.

The two above propositions are likely the most general form of necessary and sufficient conditions on min-summability one may hope for. However, under some extra conditions on the sequence σ, it is possible to unify the two Conditions of Proposition 4.1 into one single and simpler condition.

Corollary 4.3. *Let σ be a sequence of integers such that there exists $c > 1$ with the property that for all large enough values of n, $\sigma_{n+1} \ge c \cdot \sigma_n$ (think of the speed function f, see Lemma 2.5). Then W is σ-summable if and only if $\sum_n F_W^{-1}(\frac{1}{\sigma_n}) < \infty$.*

Proof. Note that, under the assumption of the corollary on the growth of σ_n, Condition (i) of Proposition 4.1 always holds, provided that $\mathbb{P}\{W > 1\} < 1$.

Let σ be a sequence satisfying the condition $\sigma_{n+1} \ge c \cdot \sigma_n$ for all n. Let $a_0 = 0$ and $a_n = F_W^{-1}(\frac{1}{\sigma_n})$ for $n \ge 1$, and suppose that $\sum_{n \ge 0} a_n < \infty$. In this case, trivially $\mathbb{P}\{W > 1\} < 1$.

We show that Condition (ii) of Proposition 4.1 holds. We have

$$\begin{aligned}
\int_0^1 \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt \quad &= \quad \int_0^{a_{n-1}} \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt + \int_{a_{n-1}}^1 \big(\mathbb{P}\{W > t\} dt\big)^{\sigma_n} \\
&\le a_{n-1} + \sum_{m=1}^{n} a_{m-1}\big((\mathbb{P}\{W > a_m\})^{\sigma_n} - (\mathbb{P}\{W > a_{m-1}\})^{\sigma_n}\big) \\
&\le a_{n-1} + \sum_{m=1}^{n} a_{m-1}(1 - 1/\sigma_m)^{\sigma_n}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_n \int_0^1 \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt &\le \sum_n a_n + \sum_m a_{m-1} \sum_{n \ge m} (1 - 1/\sigma_m)^{\sigma_n} \\
&\le \sum_n a_n + \sum_m a_{m-1} \sum_{n \ge m} (1 - 1/\sigma_m)^{c^{n-m}\sigma_m} \\
&\le \sum_n a_n + \sum_m a_{m-1} \sum_{j=0}^{\infty} e^{-c^j} \\
&= O(1) \sum_n a_n < \infty.
\end{aligned}$$

This shows that W is σ-summable.

To prove the other direction, suppose that W is σ-summable, so that by Proposition 4.1,

$$\sum_n \int_0^1 \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt < \infty.$$

Since W is σ-summable, we have $F_W(1) > 0$ and so there exists an integer N such that for $n \ge N$, $a_n \le 1$. Thus,

$$\begin{aligned}
\sum_n \int_0^1 \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt &\ge \sum_{n \ge N} \int_0^{a_n} \big(\mathbb{P}\{W > t\}\big)^{\sigma_n} dt \\
&\ge \sum_{n \ge N} \int_0^{a_n} \big(1 - \mathbb{P}\{W \le a_n\}\big)^{\sigma_n} dt \\
&= \sum_{n \ge N} \int_0^{a_n} \left(1 - \frac{1}{\sigma_n}\right)^{\sigma_n} dt \\
&= \Omega(1) \sum_{n \ge N} a_n.
\end{aligned}$$

It follows that $\sum_n a_n < \infty$ and the corollary follows. $\square$

Combining the above corollary with Theorem 2.3 and Proposition 2.2, we infer a proof of Theorem 1.1.

Examples and special cases. Here we give a family of examples of applications of Proposition 4.1. The notations are those of Proposition 4.1. (In particular, Λ_n is the minimum of σ_n copies of the weight distribution W.)

(i) If $W \geq a > 0$, then Condition (ii) of Proposition 4.1 does not hold, and so $\sum_n \Lambda_n = \infty$. (This also trivially follows from $\Lambda_n \geq a$.) This example shows that the only interesting cases occur when 0 is an accumulation point of the distribution.

(ii) If $W = 0$ with probability $p > 0$, then both the conditions of Proposition 4.1 hold if $\sum_n (1-p)^{\sigma_n} < \infty$. On the other hand, $\sum_n \Lambda_n < \infty$ implies that $\sum_n (1-p-\epsilon)^{\sigma_n} < \infty$ for every $\epsilon \in (0, p)$. This case is not of prime interest either. The case $p = 0$ with 0 being an accumulation point of W is the most interesting.

(iii) If W is uniform on $[0, 1]$, then the conditions of Proposition 4.1 are equivalent to

$$\sum_n \frac{1}{\sigma_n + 1} < \infty.$$

(iv) If W is exponential, then $\Lambda_n \overset{\mathcal{L}}{=} E/\sigma_n$, where E is exponential. The sequence Λ_n has almost surely a finite sum if and only if

$$\sum_n \frac{1}{\sigma_n} < \infty.$$

(v) For the sequence $\sigma_n = n$, assuming that there is no atom at the origin and that 0 is an accumulation point for W, it is easy to verify that $\sum_n \Lambda_n < \infty$ almost surely if and only if

$$\int_0^1 \frac{1}{\mathbb{P}\{W > t\}}\, dt < \infty.$$

(vi) For the sequence $\lambda_n \sim c^n$, with $c > 1$ a positive constant, and assuming no atom at the origin, but with 0 an accumulation point for W, it is easy to verify that $\sum_n \Lambda_n < \infty$ almost surely if and only if

$$\int_0^1 \ln\left(\frac{1}{\mathbb{P}\{W > t\}}\right) dt < \infty.$$

5 Second proof of the Equivalence Theorem

We outline here a second proof of the Equivalence Theorem. Consider the following operation on an infinite weighted tree T. Remove all those children of the root who are connected to the root by an edge of weight greater than $1/2$, then remove all vertices of the second generation which are connected to the first generation by an edge of weight greater than $1/4$, continuing this way and remove, for each $n \in \mathbb{N}$, those vertices of generation n which are connected to generation $n-1$ by an edge of weight greater than $1/2^n$. If the surviving tree contains an infinite path then we have found an infinite path of weight at most 1 in the original weighted tree.

More generally for a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, we define the trimmed version $T(a_1, a_2, a_3, \dots)$ of T to be obtained by removing, for each $n \in \mathbb{N}$, those vertices of generation n which are joined to generation $n-1$ by an edge of weight greater than a_n. If the sequence $(a_n)_{n\in\mathbb{N}}$ is summable and $T(a_1, a_2, a_3, \dots)$ contains an infinite path, then we deduce that T contains an infinite path of finite weight. In what follows we shall choose the summable sequence $(a_n)_{n\in\mathbb{N}}$ based on the offspring distribution and the weight distribution under investigation.

We stress that in this section we simply sketch a second proof of the Equivalence Theorem; the proof is complete except that we omit the proof of the Claims I and II made below. Let Z be a plump offspring distribution and let W be a weight distribution in $\mathcal{W}_{\text{MS}}(Z)$. Based on Z and W, we define a summable sequence $(a_n)_{n\in\mathbb{N}}$ such that, with positive probability, the trimmed process $T_Z^W(a_1, a_2, a_3, \dots)$ is infinite. As Z and W are arbitrary, and explosion is a 0-1 event, the Equivalence Theorem follows.

In proving that $T_Z^W(a_1, a_2, a_3, \dots)$ is infinite with positive probability, it is natural to prove the stronger statement that, with positive probability, the generation sizes of the trimmed process grow almost as fast as those of the untrimmed process. To make this statement precise, we require some more definitions. We may consider separately the branching process and the trimming process. Thus, we may continue to denote by Z_n the size of the untrimmed branching process, and, for each sequence finite sequence $(a_1, \dots, a_s)$, we may denote by $Z_n(a_1, \dots, a_s)$ the size of the n-th generation after trimming by $a_1, \dots, a_s$. For example $Z_n(a_1, \dots, a_n)$ denotes the size of the n-th generation of the trimmed process. The proof in fact shows that with positive probability

$$Z_n(a_1, \dots, a_n) \geq f(n+K) \quad \text{for all}\, n \in \mathbb{N}\,, \tag{14}$$

where f (a speed of the branching process), K (a constant (dependent on Z)) and the summable sequence $(a_n)_{n\in\mathbb{N}}$ are defined below.

We now define the speed f and the constant K. Let $\epsilon > 0$ be such that (3) holds for Z, and let m_0 be chosen such that (3) holds for all $m \geq m_0$. We may choose ϵ and m_0 such that $\epsilon \leq 1/4$ and $m_0 \geq \exp(4\epsilon^{-1})$. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(0) = 1, f(1) = m_0$ and $f(n+1) = F_Z^{-1}(1 - f(n)^{-(1+\epsilon)^{-3/4}})$ for $n \geq 1$. Let K be an integer large enough that $(1+\epsilon^{1/4})^K \geq 4\epsilon^{-1}$. We define the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n = F_W^{-1}(f(n)^{-1})\,.$$

Claim I. f is a speed of Z, and the sequence $(a_n)_{n\in\mathbb{N}}$ is summable.

That f is a speed follows from Theorem 2.3 and a variant of Lemma 2.4. It then follows from Proposition 2.2 and Corollary 4.3 that $(a_n)_{n\in\mathbb{N}}$ is summable.

Thus to complete the proof it suffices to prove that (14) holds with positive probability. Since the proof of this fact uses an inductive approach it is useful to make a stronger claim that also comments on the quantities $Z_n(a_1, \dots, a_{n-1})$.

Claim II. With positive probability,

$$\begin{aligned} & Z_n(a_1, \dots, a_{n-1}) \geq f(n+K) f(n)^{\epsilon/2} \text{ for all } n \in \mathbb{N} \\ \text{and} \qquad & Z_n(a_1, \dots, a_n) \geq f(n+K) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

One may prove the claim using an inductive approach which considers both inequalities simultaneous. Denote by p_n the probability that $Z_n(a_1,\ldots,a_{n-1}) \geq f(n+K)f(n)^{\epsilon/2}$ and $Z_n(a_1,\ldots,a_n) < f(n+K)$; denote by q_n the probability that $Z_{n-1}(a_1,\ldots,a_{n-1}) \geq f(n+K-1)$ and $Z_n(a_1,\ldots,a_{n-1}) < f(n+K)f(n)^{\epsilon/2}$. One may prove the bound $\sum(p_n+q_n) < 1$; the result then follows by a union bound.

The above two claims finish the (second) proof of the Equivalence Theorem.

6 Sharpness of the condition in the Equivalence Theorem

The main result of this article, the Equivalence Theorem, gives a sufficient condition on a distribution Z for the equality $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ to occur. This condition, that for some $\epsilon > 0$, the inequality $\mathbb{P}\{Z \geq m^{1+\epsilon}\} \geq 1/m$ holds for all sufficiently large $m \in \mathbb{N}$, demands that Z has a heavy tail, and, furthermore, that the tail is consistently heavy. This condition ensures that the generation sizes (equivalently, the speed) of the corresponding branching process are at least double exponential. Furthermore, it ensures that the rate of growth is always at least the rate associated with double exponential functions (i.e. $f(n+1) \geq f(n)^{1+\epsilon}$). It is therefore natural to ask:

(i) Could a weaker version of our condition still imply $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$?

(ii) Could a lower bound on the speed of Z alone (e.g., Z has a speed f which is at least double exponential) be sufficient to guarantee $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$?

Theorem 6.1 answers (i) in the negative (almost completely) by showing that no substantially weaker version of our condition implies $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$. Theorem 6.2 answers (ii), completely, in the negative. In a sense, these results show the Equivalence Theorem to be best possible.

Theorem 6.1. *Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function satisfying $g(m) = m^{1+o(1)}$. Then there is a distribution Z, satisfying $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$ for all $m \in N$, but for which $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$.*

Theorem 6.2. *Let $s : \mathbb{N} \to \mathbb{N}$ be any function. Then there is a function $f : \mathbb{N} \to \mathbb{N}$, satisfying $f(n) \geq s(n)$ for all $n \in \mathbb{N}$, and a distribution Z for which f is a speed, such that $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$.*

There does not seem to be an obvious intuitive way to judge, for a given distribution Z, whether the equality $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ should hold or not. So before giving our proof of Theorem 6.1, we establish a sufficient condition for the equality to fail, see Proposition 6.4 below.

We recall that a function $f : \mathbb{N} \to \mathbb{N}$ is a speed of a distribution Z if there exist $a, b \in \mathbb{N}$ such that with positive probability the bounds $Z_{n/a} \leq f(n) \leq Z_{bn}$ hold for all n. We shall say that f is a *dominating* speed if we may take $a = 1$. We shall say that f is *swift* if, for some $c > 1$, the inequality $f(n+1) > cf(n)$ holds for all $n \geq 0$. It will be useful (for technical reasons) to restrict our attention to swift dominating speeds. The following direct consequence of Corollary 4.3 and Proposition 2.2 will be useful in our proof of Proposition 6.4.

Lemma 6.3. *Let Z be a distribution with mean greater than 1, f a swift speed of Z, and W a weight distribution for which the sum $\sum_{n=1}^{\infty} F_W^{-1}(f(n)^{-1})$ is bounded. Then $W \in \mathcal{W}_{\mathrm{MS}}(Z)$.*

Proposition 6.4. *Let Z be any distribution with a swift dominating speed f satisfying*

$$\liminf_{n\to\infty} 2^n f(n) f(\lceil n/\omega(n)\rceil)^{-n/2} \;=\; 0, \tag{15}$$

for some function $\omega(n) \to \infty$ as $n \to \infty$. Then $\mathcal{W}_{\mathrm{EX}}(Z) \neq \mathcal{W}_{\mathrm{MS}}(Z)$.

Proof. We must prove the existence of a weight distribution W such that $W \in \mathcal{W}_{\mathrm{MS}}(Z)$ but $W \notin \mathcal{W}_{\mathrm{EX}}(Z)$. Before defining W, we first define some sequences on which its definition will be based. From our assumption on f, there exists an increasing sequence n_i such that

$$\lim_{i\to\infty} 2^{n_i} f(n_i) f(\lceil n_i/\omega(n_i)\rceil)^{-n_i/2} \;=\; 0. \tag{16}$$

Let us define the sequence ω_i by $\omega_i = \omega(n_i)$ and the sequence β_i by $\beta_i = \sqrt{\omega_i}$. We note that $\beta_i \to \infty$ as $i \to \infty$, and so we may choose a subsequence β_{i_j} with the property that $\beta_{i_j} \geq 2^j$ for each $j \geq 1$. Finally, set $m_i := \lceil n_i/\omega_i \rceil$. We now define the weight distribution W to satisfy

$$\mathbb{P}\left\{W < \frac{1}{\beta_{i_j} m_{i_j}}\right\} \;=\; \frac{1}{f(m_{i_j})} \qquad \text{for all } j \geq 1,$$

by placing probability mass $f(m_{i_j})^{-1} - \sum_{j'>j} f(m_{i_{j'}})^{-1}$ at position $1/\beta_{i_{j+1}} m_{i_{j+1}}$ for each $j \geq 1$, and probability mass $1 - \sum_{j' \geq 1} f(m_{i_{j'}})^{-1}$ at 1.

We first observe that $W \in \mathcal{W}_{\mathrm{MS}}(Z)$. Indeed, this follows immediately from Lemma 6.3 and the observation that

$$\sum_{n\geq 1} F_W^{-1}(f(n)^{-1}) \;\leq\; \sum_{j\geq 1} m_{i_j} \cdot \frac{1}{\beta_{i_j} m_{i_j}} \;\leq\; \sum_{j\geq 1} \frac{1}{\beta_{i_j}} \;\leq\; \sum_{j\geq 1} \frac{1}{2^j} \;=\; 1.$$

We now observe that $W \notin \mathcal{W}_{\mathrm{EX}}(Z)$. We must prove that $\mathbb{P}\{E\} < 1$, where E denotes the event of an infinite path of finite weight. Let G be the event G that $Z_n \leq f(n)$ for all $n \in \mathbb{N}$; since since f is a dominating speed of Z, G has positive probability. Thus it suffices to prove that $\mathbb{P}\{E \mid G\} = 0$.

Let A_j be the event that there exists a path from the root to generation n_{i_j} of weight less than $\beta_{i_j}/2$. The event E may occur only if A_j occurs for all sufficiently large j, so it suffices to prove that $\mathbb{P}\{A_j \mid G\} \to 0$ as $j \to \infty$.

For the event A_j to occur there must exist a path from the root to generation n_{i_j} at least half of whose edges have weight less than β_{i_j}/n_{i_j}. Since under event G there are at most $f(n_{i_j})$ such paths, and for each path there are less than $2^{n_{i_j}}$ choices for a subset of half its edges, we have

$$\mathbb{P}\{A_j \mid G\} \leq 2^{n_{i_j}} f(n_{i_j}) \left(\mathbb{P}\{W < \beta_{i_j}/n_{i_j}\}\right)^{n_{i_j}/2}.$$

Since

$$\mathbb{P}\{W < \beta_{i_j}/n_{i_j}\} = \mathbb{P}\{W < 1/(\beta_{i_j} m_{i_j})\} = 1/f(m_{i_j}),$$

it follows from (16) that $\mathbb{P}\{A_j \mid G\} \to 0$ as required. $\square$

Proof of Theorem 6.1. Let g be any increasing function satisfying the condition of the theorem, i.e., $g(m) = m^{1+o(1)}$. We define a distribution Z satisfying $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$ for all $m \in \mathbb{N}$, which has a swift dominating speed f satisfying $\liminf_{n\to\infty} 2^n f(n) f(\lceil n^{1/2} \rceil)^{-n/2} = 0$; the proof is then complete by Proposition 6.4.

There is a sense in which it is difficult to achieve these two objectives simultaneously. The first asks that Z has a sufficiently heavy tail, while the second would seem to get more likely to occur if the tail of Z were less heavy. Our approach to achieving the objectives simultaneously is to define Z to have a heavy, but not at all smooth, tail. In the resulting Galton-Watson branching process the growth of generation sizes does not at all resemble a smooth fast growing function (such as a double exponential), but instead consists of a number of periods of exponential growth, each period much longer than all proceeding periods, and with a multiplicative factor very much larger (in fact the lengths will be $(2n_i)_{i\geq 1}$ and the multiplicative factors $(m_i)_{i\geq 1}$; these sequences are defined below)

Define $n_i = 10^{10^i}$ for each $i \geq 1$, and $\epsilon_i = 1/10n_i = 10^{-(10^i+1)}$. As $g(m)) = m^{1+o(1)}$, there exists, for each ϵ_i, a natural number m_i such that $g(m) \leq m^{1+\epsilon_i}$ for all $m \geq m_i^{1/2}$. Furthermore, we may choose $(m_i)_{i\in\mathbb{N}}$ to in addition satisfy

$$m_i \geq 16 n_i^2 M_{i-1}^2 \quad \text{for all} \quad i \geq 1, \tag{17}$$

where $M_0 = 1$ and $M_j := \prod_{i=1}^{j} m_i^{2n_i}$ for $j \geq 1$. Next define sequences $(N_j)_{j\in\mathbb{N}}$ and $(L_j)_{j\in\mathbb{N}}$ by

$$N_j := \sum_{i=1}^{j} n_i \quad \text{and} \quad L_j := m_j \prod_{i=1}^{j-1} m_i^{2n_i}.$$

As we mentioned above, we shall define the distribution Z so that the growth of generation sizes of T_Z consists of a number of periods of exponential growth, each period much longer than all proceeding periods, and with multiplicative factor very much larger. (The jth period of growth will have length (approximately) $2n_j$ and multiplicative factor m_j.) In this context L_j is approximately the generation size at the start of this jth period of growth (in fact after the first step of this period) and M_j the generation size when it ends (i.e., at the point at which we shall switch into the next, faster, period of growth). One may observe that $L_j = m_j M_{j-1}$; note however that L_j is much larger than M_{j-1}, since (17) implies that m_j is already much larger.

Define the distribution Z by

$$\begin{aligned}
\mathbb{P}\{Z \geq L_1\} &= 1; \\
\mathbb{P}\{Z \geq m^{1+\epsilon_i}\} &= \frac{1}{m}, \qquad L_i^{1/(1+\epsilon_i)} < m \leq M_i,\ i \geq 1; \\
\mathbb{P}\{Z \geq L_{i+1}\} &= \frac{1}{M_i}, \qquad i \geq 1.
\end{aligned}$$

It is easily verified that this distribution satisfies $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$ for all $m \in \mathbb{N}$. Now define the function $f : \mathbb{N} \to \mathbb{N}$ (which will be a speed for Z) by

$$f(n) = L_{i+1} m_{i+1}^{2(n-N_i)-1} \quad \text{with } i \text{ chosen so that } N_i < n \leq N_{i+1}.$$

It is also quite easily verified that f satisfies (15), using $\omega(n) = n^{1/2}$. In particular, we observe that $f(n_i) \leq L_i m_i^{2n_i}$, and, since $\lceil n_i^{1/2} \rceil - N_{i-1} \geq n_{i-1}$, we have that $f(\lceil n_i^{1/2} \rceil)^{n_i/2} \geq L_i m_i^{n_{i-1} n_i}$.

It is also easily observed that f is swift. Thus, in light of Proposition 6.4, all that is required to complete the proof is to demonstrate that f is a dominating speed of Z. Though it is conceptually straightforward, the proof is rather long; we stress that it is really just a technical detail.

We prove that with positive probability the bounds $Z_n \le f(n) \le Z_{4n}$ hold for all $n \in \mathbb{N}$. Let E be the event that $Z_n > f(n)$ for some n, and let F be the event that $Z_{4n} < f(n)$ for some n. Let us subdivide these events by the minimum n for which the required inequality fails. Let E_n to be the event that n is minimal such that $Z_n > f(n)$, and F_n the event that n is minimal such that $Z_{4n} < f(n)$. We will show that $\sum_{n\ge1} E_n \le 1/4$ and $\sum_{n\ge1} F_n \le 1/4$, which will complete the proof.

We have stated that our example is designed to exhibit a number of periods of exponential growth. Once the number of nodes of a given generation is much larger than M_{i-1}, it is clear that, from this point on, the growth should always be at least geometric (i.e., exponential) with multiple m_i. Indeed, among $m \gg M_{i-1}$ nodes, one expects about m/M_{i-1} to have $L_i = m_i M_{i-1}$ children. Considering these children alone we see that the size of the next generation should be at least m_i times as large.

Our bound on the probability of the event F is therefore relatively straightforward, requiring us to formalize the above statement. The bound on the probability of E is more difficult as we are required to control all ways in which the process could grow faster.

Claim. $\mathbb{P}\{E\} \le 1/4$.

Proof. We shall define two sequences $p_{i,j,k}$ and q_i of probabilities, corresponding to the probabilities of certain unlikely events (events that would cause faster than expected growth). We then prove a bound on the probability of E based on the $p_{i,j,k}$ and q_i, specifically that this probability is at most their sum. It then suffices to bound by $1/4$ the sum $\sum_{i,j,k} p_{i,j,k} + \sum_i q_i$.

For each triple $i,j,k \in \mathbb{N}_0$ such that $i \ge 1$, $1 \le j \le n_i - 1$ and $0 \le k \le 4j$, we define $p_{i,j,k}$ to be the probability that amongst $M_{i-1}m_i^{2j}$ independent copies of Z, at least $M_{i-1}m_i^{k/2}$ exceed $M_{i-1}m_i^{2j+1-k/2}$. We define q_1 to be the probability that $Z \ge m_1^2$, and, for $i \ge 2$, we define q_i to be the probability that amongst M_{i-1} copies of Z, at least one of them exceeds $M_{i-1}m_i^{3/2}$.

We prove the bound

$$\mathbb{P}\{E\} = \sum_{n\ge1} \mathbb{P}\{E_n\} \le \sum_{i,j,k} p_{i,j,k} + \sum_i q_i.$$

Notice that for the event $E_{N_{i-1}+1}$ to occur, we must have

$$Z_{N_{i-1}} \le f(N_{i-1}) = M_{i-1} \quad \text{and} \quad Z_{N_{i-1}+1} > f(N_{i-1}+1) = M_{i-1}m_i^2.$$

This in turn implies that at least one of the nodes in generation N_{i-1} has more than $M_{i-1}m_i^{3/2}$ children (as $M_{i-1} \le m_i^{1/2}$, see Condition 17). Thus we may bound for each i the probability of the event $E_{N_{i-1}+1}$ by q_i.

Next, for n of the form $N_{i-1}+j+1$ for some $i \in \mathbb{N}$ and $1 \le j \le n_i - 1$, we note that the occurrence of E_n implies that

$$Z_{n-1} \le M_{i-1}m_i^{2j} \quad \text{and} \quad Z_n > M_{i-1}m_i^{2j+2}.$$

It follows that for some $0 \le k \le 4j$, there are at least $M_{i-1}m_i^{k/2}$ nodes of generation $n-1$ with more than $M_{i-1}m_i^{2j+1-k/2}$ children. Indeed, if this were not the case, then we would have

$$\begin{aligned} Z_n &\le \sum_{k=0}^{4j}(M_{i-1}m_i^{k/2})(M_{i-1}m_i^{2j+3/2-k/2}) \\ &= (4j+1)M_{i-1}^2 m_i^{2j+3/2} \\ &\le M_{i-1}m_i^{2j+2} \qquad \text{(since } (4j+1)M_{i-1} \le 4n_iM_{i-1} \le m_i^{1/2}\text{)}. \end{aligned}$$

It easily follows that $\mathbb{P}\{E_n\} \le \sum_{0\le k\le 4j} p_{i,j,k}$.

We now prove the bound $\sum_{i,j,k} p_{i,j,k} + \sum_i q_i \le 1/4$. By the bounds (17) it suffices to prove for each triple $i,j,k \in \mathbb{N}_0$ with $i \ge 1$, $1 \le j \le n_i - 1$ and $0 \le k \le 4j$, that

$$p_{i,j,k} \le (m_i/e^2)^{-M_{i-1}m_i^{k/2}/2} \tag{18}$$

and

$$q_i \le \frac{M_{i-1}}{m_i}.$$

The bound on q_i is trivial; since $1/(1+\epsilon_i) \ge 2/3$, it follows that

$$\mathbb{P}\{Z \ge M_{i-1}m_i^{3/2}\} = (M_{i-1}m_i^{3/2})^{-1/(1+\epsilon_i)} \le m_i^{-1}.$$

We bound the probability $p_{i,j,k}$ (that amongst $M_{i-1}m_i^{2j}$ independent copies of Z at least $M_{i-1}m_i^{k/2}$ exceed $M_{i-1}m_i^{2j+1-k/2}$) using a union bound. By the familiar estimate $\binom{s}{t} \le (es/t)^t$, the number of choices of the set of $M_{i-1}m_i^{k/2}$ copies is

$$\binom{M_{i-1}m_i^{2j}}{M_{i-1}m_i^{k/2}} \le (em_i^{2j-k/2})^{M_{i-1}m_i^{k/2}}.$$

For each copy of Z we have

$$\mathbb{P}\{Z > M_{i-1}m_i^{2j+1-k/2}\} = (M_{i-1}m_i^{2j+1-k/2})^{-1/(1+\epsilon_i)} \le m_i^{-(2j+1/2-k/2)},$$

where for the final inequality we have used that $\epsilon_i = 1/(10n_i)$ and (since $2j+1/2-k/2 \le 2n_i$)

$$2j+1-k/2 = 2j+1/2-k/2+1/2 \ge (2j+1/2-k/2)(1+1/(4n_i)).$$

Thus the probability that a given set of $M_{i-1}m_i^{k/2}$ copies of Z all exceed $M_{i-1}m_i^{2j+1-k/2}$ is at most

$$m_i^{-(2j+1/2-k/2)M_{i-1}m_i^{k/2}},$$

and (18) now follows by a union bound. $\square$

Claim. $\sum_{n\ge1}\mathbb{P}\{F_n\} \le 1/4$.

Proof. Our approach is similar to that used in the previous proof. For $i \ge 1$ and $2 \le j \le 4n_i$, we define $p_{i,j}$ to be the probability that from a collection of $M_{i-1}m_i^{j/2}$ copies of Z, fewer than

$M_{i-1}m_i^{j/2-1/2}$ exceed m_i. For each $i \geq 1$, we define q_i to be the probability that the maximum of $M_i m_i^{1/2}$ copies of Z is less than L_{i+1}. We prove for n of the form $n = N_i + 1$ that

$$\mathbb{P}\{F_n\} \leq p_{i,4n_i} + q_i + p_{i+1,2} + p_{i+1,3},$$

and for n of the form $n = N_i + k$, $k = 2, \dots n_{i+1}$, that

$$\mathbb{P}\{F_n\} \leq p_{i+1,4k-4} + p_{i+1,4k-3} + p_{i+1,4k-2} + p_{i+1,4k-1}.$$

It will then suffice to bound by 1/4 the sum $\sum_{i,j} p_{i,j} + \sum_i q_i$. For $n = N_i + k$, $k = 2, \dots, n_{i+1}$, if the event F_n occurs then $Z_{4n-4} \geq f(n-1) = M_i m_{i+1}^{2k-2}$ and $Z_{4n} < f(n) = M_i m_{i+1}^{2k}$. The required bound now follows, as the probability for a given $0 \leq l \leq 3$ that l is minimal such that $Z_{4n-l} < M_i m_{i+1}^{2k-l/2}$ is at most $p_{i+1,4k-l-1}$. The case $n = N_i + 1$ is similar, differing only in that we do not consider the events $Z_{4n-l} < M_i m_{i+1}^{2k-l/2}$ for $0 \leq l \leq 3$, but rather the events $Z_{4n-3} < M_i m_i^{1/2}$, $Z_{4n-2} < L_{i+1}$, $Z_{4n-1} < L_{i+1} m_i^{1/2}$ and $Z_{4n} < L_{i+1} m_i$.

Finally we prove the bound $\sum_{i,j} p_{i,j} + \sum_i q_i < 1/4$. It is trivial, using the inequality $(1-p)^n \leq e^{-pn}$, that $q_i \leq \exp(-\sqrt{m_i})$. To bound $p_{i,j}$ we first note that $\mathbb{P}\{Z > m_i\} \geq 1/M_{i-1}$, so from a collection of $M_{i-1}m_i^{j/2}$ copies of Z the distribution for the number exceeding m_i is $\mathrm{Bin}(M_{i-1}m_i^{j/2}, 1/M_i)$. Since this binomial has expected value $m_i^{j/2} \geq 2M_{i-1}m_i^{j/2-1/2}$, an application of Chernoff's inequality yields

$$p_{i,j} \leq \exp\left(\frac{-m_i^{j/2}}{8}\right).$$

$\square$

The proof of Theorem 6.1 is now complete. $\square$

The proof of Theorem 6.2 is essentially identical to the above. The only change required is that the following extra condition should be included in (17):

$$m_i \geq \max_{n \leq n_i} s(n) \quad i \geq 1.$$

This ensures that the inequality $f(n) \geq s(n)$ holds for all $n \in \mathbb{N}$. Since the proofs that f is a speed of Z and that $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$ are unaffected by this change, Theorem 6.2 does indeed follow.

7 Limit theorem in the case of no explosion

So far we only considered the appearance of the event of explosion. In this section, we consider the case of weight distributions for a heavy-tailed branching random walk for which explosion does not happen, and obtain a precise limit theorem for the minimum displacement M_n under some quite strong (smoothness) assumption on the tails of Z. To explain this, let Z be a plump random variable, and denote by $G_Z(.)$ the moment generating function of Z as before. Note

that

$$\begin{aligned}
K_Z(s) = 1 - G_Z(1-s) &= \sum_{k=0}^{\infty}\Big(\mathbb{P}\{Z=k\} - (1-s)^k\mathbb{P}\{Z=k\}\Big) \\
&= s\sum_{k=1}^{\infty}\mathbb{P}\{Z=k\}(1+\cdots+(1-s)^{k-1}) \\
&= s\Big(1-\mathbb{P}\{Z=0\} + \sum_{k=1}^{\infty}(1-s)^k(1-F_Z(k))\Big). \qquad (19)
\end{aligned}$$

Consider now the smoothness Condition 5 on Z:

$$1 - F_Z(k) = k^{-\eta}\ell(k),$$

for some function ℓ which is continuous-bounded-and-non-zero at infinity. In particular, note that one can define $\ell(\infty) \neq 0, \infty$. Using Equation (19) and applying a Tauberian theorem (see for example Feller [18, XIII. 5, Thm. 5]), we see that Condition 5 is equivalent to the condition

$$K_Z(s) \sim a\, s^{\eta}\ell(\frac{1}{s}) \qquad (\star)$$

near $s = 0$ for some $a > 0$ (indeed, $a = \Gamma(1-\eta)$). This in particular implies that Z is plump and

$$F_Z^{-1}(1 - \frac{1}{m}) = m^{1+\epsilon}\tilde{\ell}(m), \qquad (\star\star)$$

for a slowly growing function $\tilde{\ell}$ and $1 + \epsilon = \eta^{-1}$. We have

Theorem 7.1. *Let Z be an offspring distribution satisfying ($\star$). Let W be a nonnegative weight distribution and assume that $W \notin \mathcal{W}_{\text{EX}}(Z)$. Conditional on the survival of the Galton-Watson process,*

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{n} F_W^{-1}\big(\frac{1}{h(k)}\big)} = 1.$$

Here $h(k) = \exp\big((1+\epsilon)^k\big)$, where ϵ is as in ($\star\star$) and $\eta = (1+\epsilon)^{-1}$ as in ($\star$).

The proof will essentially use the algorithm we presented in Section 3. However, we first need to obtain a more precise information on the speed of the Galton-Watson tree under Condition ($\star$).

Definition 7.1 (Additive speed)**.** An increasing function $h : \mathbb{N} \to \mathbb{R}^+$ is an *additive* speed for a Galton-Watson offspring distribution Z if the probability of the increasing events E_r defined as

$$E_r := \Big\{ \quad h(n-r) \le Z_n \le h(n+r) \qquad \text{for all large enough } n \quad \Big\}$$

tend to one as r goes to infinity conditional on survival.

Lemma 7.2. *Let Z be an offspring distribution satisfying Condition ($\star$). Then the function $h : \mathbb{N} \to \mathbb{R}^+$ defined by $h(n) = \exp\big((1+\epsilon)^k\big)$ is an additive speed for Z.*

Proof of Theorem 7.1. Since $h(n)$ is an additive speed for Z, we obtain by Lemma 7.2 that conditional on survival,

$$\lim_{r\to\infty} \mathbb{P}\{E_r\} = 1.$$

Fix the integer r and suppose the event E_r holds. This means $Z_n \le h(n+r)$ for large enough n. This implies that the minimum of level n is at least $F_W^{-1}(\frac{1}{h(n+r)})$ for all large enough n. Since by our Equivalence Theorem we have a.s. $\sum F_W^{-1}(1/h(n)) = \infty$, we obtain

$$\liminf_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(k)}\big)} = \liminf_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(k+r)}\big)} \ge 1,$$

on E_r. We infer that on the union of E_r, i.e., on the event of non-exctinction, we have

$$\liminf_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(k)}\big)} \ge 1.$$

We now show that on the union of E_r, we have

$$\limsup_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(k)}\big)} \le 1.$$

This will finish the proof of the theorem above.

It will be enough to show this on each E_r. In addition, we can also fix an n_0 and suppose that for all $n \ge n_0$, we have $Z_n \ge h(n-r)$ (and then make n_0 tend to infinity). Fix a small $\delta > 0$. One can now apply a variant of the algorithm of Section 3, by modifying α to $(1+\epsilon)^{-\delta}$, started at some large $N > n_0$, and show that w.h.p., as N goes to infinity, we have for all $n \ge N$, $X_n \ge h((1-\delta)n)$ (this follows from a variant of the inequalities (12) and (13)). In addition, given the double exponential growth of $h(n)$, a union bound argument shows that we can assume with height probability that for large enough n, the weight of the n-th edge on the path constructed in the algorithm is bounded above by $F_W^{-1}(1/h((1-2\delta)n))$. Applying now the Equivalence Theorem, since both M_n and $\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h((1-2\delta)k)}\big)$ tend to infinity, we obtain that

$$\limsup_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h((1-2\delta)k)}\big)} \le 1.$$

Since this holds for any small enough $\delta > 0$, and since the function $F_W^{-1}(1/m)$ is a decreasing function of m, a simple argument shows that

$$\limsup_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(k)}\big)} = \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\big(\frac{1}{h(\,(1-2\delta)k\,)}\big)} \le 1.$$

The theorem follows. ∎

Proof of Lemma 7.2. Under some extra conditions on ℓ as in Seneta [31] or [32], a combination of the results of Darling [14] and Cohn [13] with the above mentioned results of Seneta [31, 32] ensures the existence of a limiting random variable V such that

$$(1+\epsilon)^{-n} \log(Z_n + 1) \to V \qquad \text{almost surely,}$$

for V having a strictly increasing continuous distribution v, $V > 0$ a.s. on the set of non-extinction of the process, and $v(0+) = q$, where q is the extinction probability of the Galton-Watson process. In the general case of a function ℓ continuous bounded and non-zero at infinity, the above limit theorem still holds as we now briefly explain by following closely Bramson's strategy in [10]. Define $\alpha = 1+\epsilon = \eta^{-1}$. The general idea in proving such a limit theorem is to prove first the convergence of the sequences $K^{(n)}\big(\exp(-\alpha^n s)\big)$ uniformly on compact sets. Here, $K^{(n)}(\cdot) = K_Z^{(n)}(\cdot) = K_{Z_n}(\cdot)$ is the n-times composition of K_Z (and K_Z is as in Equation (19)). For this, define

$$H(s) := -\log K\big(\exp(-s)\big),$$

and notice that $H^{(n)}(s) = -\log K^{(n)}\big(\exp(-s)\big)$, so that we are left to prove the convergence of the sequence $H^{(n)}(\alpha^n s)$ as n goes to infinity, for $s \geq 0$.

By an abuse of the notation (from Condition $(\star)$), assume that $K_Z(s) = s^\eta \ell(\frac{1}{s})$ for a function ℓ continuous bounded and non-zero at infinity, and define

$$L(s) = -\log \ell(\exp(s)).$$

By the assumptions on ℓ, it follows that L is continuous at infinity and $L(\infty) \neq \pm\infty$, and so for each $a > 0$, there is an N_a such that for s_1 and s_2 larger than N_a, we have $|L(s_1) - L(s_2)| \leq a$. A simple induction shows that

$$H^{(m)}(\alpha^m s) = s + \sum_{k=1}^{m} \frac{1}{\alpha^{m-k}}\,(-1)^k L\Big(\,H^{(k-1)}(\alpha^{m-k+1}s)\,\Big). \tag{20}$$

By the definition of H, one can easily verify that H is 1-Lipschitz, i.e.,

$$\text{for any two } s_1, s_2 \geq 0, \qquad |H(s_1) - H(s_2)| \leq |s_1 - s_2|.$$

We now show that the sequence $\{H^{(n)}(\alpha^n s),\ n \in \mathbb{N}\}$ is Cauchy, proving the point-wise convergence. The same argument shows that the sequence is uniformly Cauchy on compact intervals of $[0,\infty)$ concluding the proof of the uniform convergence.

Fix a large $m \in \mathbb{N}$ and note that replacing s by $\alpha^n s$ in (20), we get

$$H^{(m)}(\alpha^{n+m} s) = \alpha^n s + \sum_{k=1}^{m} \frac{1}{\alpha^{m-k}}\,(-1)^k L\Big(\,H^{(k-1)}(\alpha^{m-k+1+n}s)\,\Big).$$

We claim that as n goes to infinity each term $H^{(k-1)}(\alpha^{m-k+1+n}s)$ tends to infinity. Indeed, more precisely, the rate of convergence to infinity of this term is as $\alpha^{n+m-2k+2}s + O(1)$; this can be shown by a simple induction from (20), using the bounded continuity of L at infinity.

For two fixed m and M, we have

$$\begin{aligned}\Big|H^{(m)}(\alpha^{n+m}s) - H^{(M)}(\alpha^{n+M}s)\Big| = \Big|&\sum_{k=1}^{m} \frac{1}{\alpha^{m-k}}\,(-1)^k L\Big(H^{(k-1)}(\alpha^{m-k+1+n}s)\Big) \\ &- \sum_{k=1}^{M} \frac{1}{\alpha^{M-k}}\,(-1)^k L\Big(H^{(k-1)}(\alpha^{M-k+1+n}s)\Big)\Big|.\end{aligned}$$

For n large enough, we can assume that each term $L\big(H^{(k-1)}(\alpha^{(m-k+1+n)}s)\big)$ differs from $L(\infty)$ by an arbitrary small positive number a. It follows then

$$\left| H^{(m)}(\alpha^{n+m}s) - H^{(M)}(\alpha^{n+M}s) \right| \ \le\ a \left[\sum_{k=1}^{m} \frac{1}{\alpha^{m-k}} + \sum_{k=1}^{M} \frac{1}{\alpha^{M-k}} \right] + \left| \sum_{k=1}^{m} \frac{1}{\alpha^{m-k}} (-1)^k L(\infty) - \sum_{k=1}^{M} \frac{1}{\alpha^{M-k}} (-1)^k L(\infty) \right|.$$

Since $\alpha > 0$ and $L(\infty) < \infty$, and a can be chosen arbitrarily small, obviously the right term of the above inequality can be made arbitrarily small, provided that n is sufficiently large and the constants m and M are large enough. We conclude that for any $a > 0$, there exist integer constants N_a and M_a such that

$$\begin{aligned} \left|H^{(n+m)}(\alpha^{n+m}s) - H^{(n+M)}(\alpha^{n+M}s)\right| \ &\le\ \left|H^{(m)}(\alpha^{n+m}s) - H^{(M)}(\alpha^{n+M}s)\right| \\ &\le\ a \end{aligned}$$

for any n larger than N_a, provided that m and M are larger than M_a. This shows that the sequence is Cauchy. In the same way, we can easily prove that the sequence is uniformly Cauchy on compact subsets of $[0, \infty)$. This shows the existence of a continuous limit w for the sequence $H^{(n)}(\alpha^n s)$.

We now show that w is strictly increasing and $w(\infty) = \infty$. For this, note that for $s_1 < s_2$, the above arguments show that for large enough m and n, one has $H^{(m)}(\alpha^{n+m}s_i) = \alpha^n s_i + O(1)$. In particular for n large enough constant and for all m, $H^{(m)}(\alpha^{n+m}s_2) - H^{(m)}(\alpha^{n+m}s_1) > \frac{1}{2}\alpha^n(s_2 - s_1)$. Since H is itself strictly increasing, and so $H^{(n)}$ is, one conclude that the limit w is strictly increasing. A similar argument shows that $w(\infty) = \infty$.

Finally, we observe that $w(0^+) = -\log(1-q)$. This follows from a simple fixed point argument: fix an $s > 0$ and note that

$$\begin{aligned} w(0+) = \lim_{m\to\infty} w(\alpha^{-m}s) &= \lim_{m\to\infty}\lim_{n\to\infty} H^{(m)}H^{(n-m)}(\alpha^{n-m}s) \\ &= \lim_{m\to\infty} H^{(m)}(w(s)), \end{aligned}$$

by the continuity of $H^{(m)}$ for each fixed m.

Since $H^{(m)}(w(s)) = -\log K_Z^{(m)}(\exp(-w(s)))$ and $w(s) \ge 0$, it follows easily that for each $s > 0$, when m goes to infinity, $H^{(m)}(w(s))$ tends to the unique finite fixed point of H. This is $-\log(1-q)$, a consequence of the corresponding statement for $K^{(m)}$ given that the unique fixed point of K_Z in $(0,1)$ is $1-q$.

These then allow us to conclude the proof of the above convergence result by first proceeding as in Darling [14] to obtain the convergence in distribution, and next by applying the result of Cohn [13] to obtain the almost sure convergence.

To conclude the proof of the lemma, note that for two constants $\delta, \Delta > 0$, $\delta < \Delta$, the event

$$E_{\delta,\Delta} := \left\{ \delta(1+\epsilon)^n \le \log(Z_n+1) \le \Delta(1+\epsilon)^n \quad \text{for large enough } n \right\}$$

happens with a probability tending to $1-q$ as $\delta \to 0$ and $\Delta \to \infty$. For two fixed constants δ and Δ, we have for r large enough, $(1+\epsilon)^{-r} \le \delta$ and $(1+\epsilon)^r \ge \Delta$. This shows that the event $E_{\delta,\Delta}$ is contained in the event E_r for r sufficiently large, and the lemma follows. $\square$

8 Conclusion

We have proved the equivalence of $\mathcal{W}_{\text{EX}}(Z)$ and $\mathcal{W}_{\text{MS}}(Z)$ for plump offspring distributions Z, and shown that the plumpness condition is essentially best possible, in terms of conditions of the form $F_Z(1-1/m) \geq g(m)$. However, this is very far from being a characterization of all offspring distributions for which explosion and min-summability are equivalent. For example, a simple adaptation of the proof of the Equivalence Theorem shows that $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ for Z defined by

$$\mathbb{P}\left\{Z \geq m \exp\left(\exp\left(\log\log m - \sqrt{\log\log m} + \tfrac{1}{2}\log\log\log m\right)\right)\right\} = \frac{1}{m}.$$

The function

$$f(n) = e^{e^{\log^2 n}}$$

is a speed of Z. This illustrates that the equivalence can occur for distributions with speeds very much slower than doubly exponential. By contrast, any plump distribution has a speed that grows at least as fast as a double exponential.

We remark that the above example is extremely close to best possible. It follows from Proposition 6.4 that the equivalence cannot hold for an offspring distribution which has a speed of the form

$$f(n) = e^{e^{o(\log^2 n)}}.$$

We do not know how general the equivalence of $\mathcal{W}_{\text{EX}}(Z)$ and $\mathcal{W}_{\text{MS}}(Z)$ should be when Z has speed slower than doubly exponential. Obtaining a complete characterization of offspring distributions where equivalence occurs remains an interesting open question.

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